One-ended spanning trees and generic combinatorics

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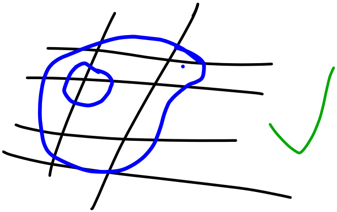
January 31, 2022

Based on joint work with Poulin and Zomback

Classical results

Throughout G will be a graph with bounded (finite) maximum degree $\Delta(G)$.

- A connected graph is **one-ended** if for every finite $F \subset V(G)$ the induced subgraph on $V(G) \setminus F$ has one infinite component.



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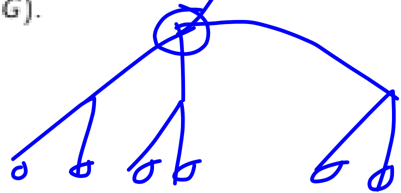

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Can we find definable analogs of the above?

Borel graphs and Baire measure

Fix from now on a Polish space (X, τ).

- A graph G with $V(G) = X$ is a **Borel graph** if $E(G) \subset X^2$ is Borel.
- A subset of $A \subseteq X$ is **nowhere dense** if $\overline{A}$ has empty interior, it is **meagre** if it is a countable union of nowhere dense sets, and it is **Baire measurable** if there is an open set U and meagre set M with $A = O \triangle M$.
- We say that G is one-ended if each of its connected components is.
- We say that G **admits a one-ended spanning tree generically** if there is a G-invariant comeagre Borel set X' and a Borel $T \subseteq G|_{X'}$ such that T is acyclic, one-ended, and spans each G component that it meats.

Known results

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- (Timar '19. Conley, Gaboriau, Marks, Tucker-Drob '21) Every one-ended Borel graph has a one-ended spanning tree a.e.

  pmp, hyp.
  - (B., Kun, Sabok '21) This is useful for showing that d-regular bipartite graphs have Borel perfect matchings a.e., and that $2d$-regular graphs admit Borel balanced orientations.

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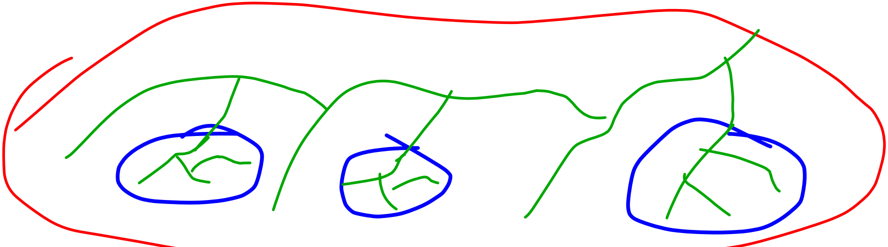
    - This is *very* useful for showing that d-regular bipartite graphs have Borel perfect matching generically, and $2d$-regular graphs have Borel balanced orientations generically.

toasts

Definition

A borel family of sets $\mathcal{T} \subset V(G)^{<\infty}$ is a **toast** if it satisfies properties (1) and (2) of the below definition, and it is a **connected toast** if it also satisfies property 3:

1. $\bigcup_{K \in \mathcal{T}} E(K) = E(G)$,
2. for every pair $K, L \in \mathcal{T}$ either $(N(K) \cup K) \cap L = \emptyset$ or $K \cup N(K) \subseteq L$, or $L \cup N(L) \subseteq K$,
3. for every $K \in \mathcal{T}$ the induced subgraph on $K \setminus \bigcup_{K \supsetneq L \in \mathcal{T}} L$ is connected.



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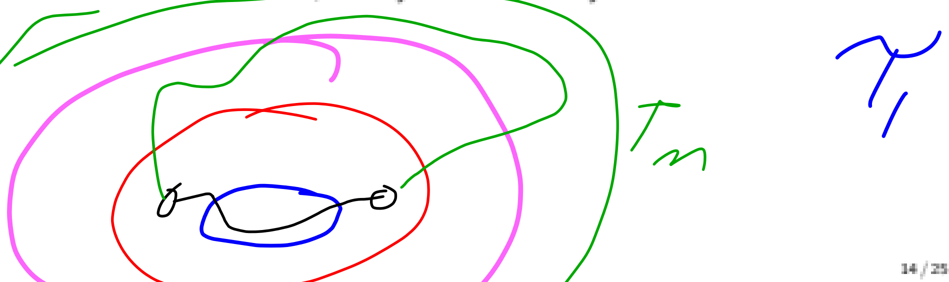
- (Brandt, Chang, Grebik, Grunau, Rozhon, Vidnyanszky 21') Every bounded degree Borel graph admits a toast generically.
- (B., Kun, Sabok '21) Every one-ended Borel graph admits a connected toast a.e.

connected toasts generically

Theorem (B., Poulin, Zomback '22+)

Every one-ended bounded degree Borel graph admits a connected toast generically.

- Fix a toast $\mathcal{T}$ and order elements of $\mathcal{T}$ by inclusion, where $\mathcal{T}_1$ is the set of minimal elements, etc.
- For every $L \in \mathcal{T}_1$ there is an $m \in \omega$ and $L \subset K \in \mathcal{T}_{<m}$ such that $K \setminus L$ is connected. Here, we say L is covered by level m.



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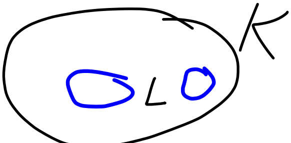
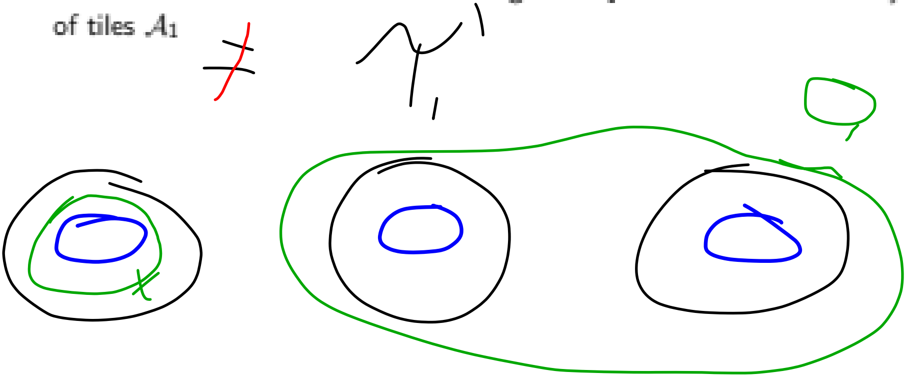
- For some m $\{x \in L \in \mathcal{T}_1 : \text{ L is covered by level m}\}$ is non-meagre in U_1. Call the family of such L tiles $\mathcal{A}'$



toast proof continued

- By the php we can choose one $\mathcal{A}_1'$ tile from each maximal $T_{<m}$ tile so that the chosen tiles are still non-meagre in U_1. Call the chosen family of tiles $\mathcal{A}_1$



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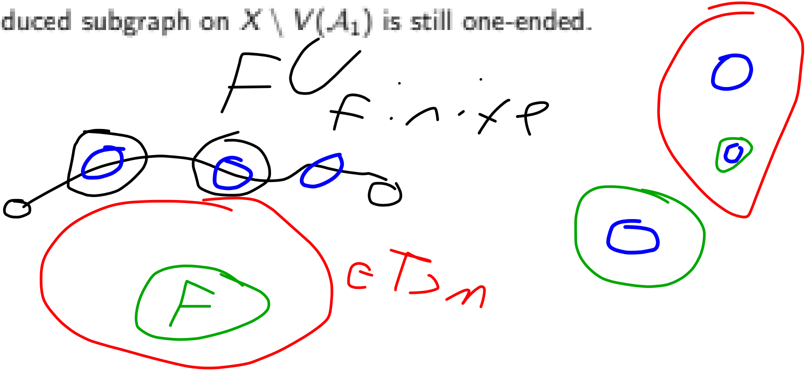
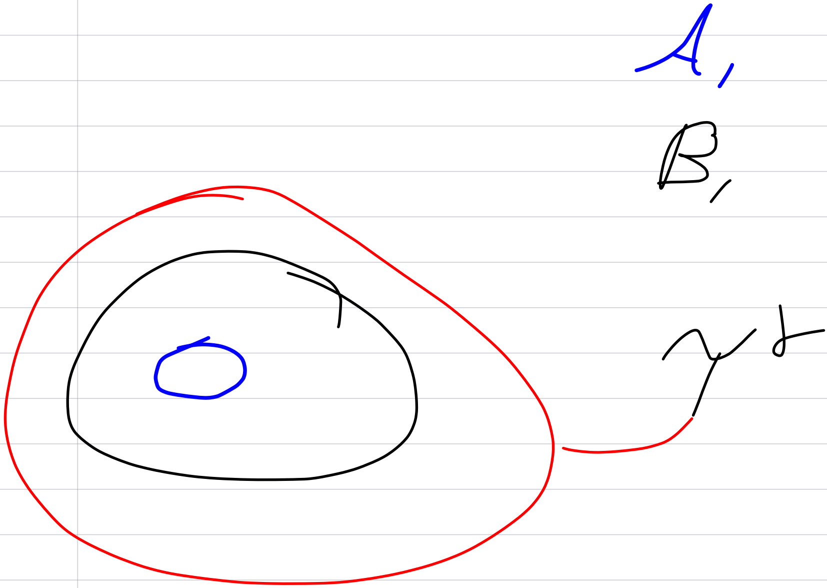
- the induced subgraph on $X \setminus V(\mathcal{A}_1)$ is still one-ended.

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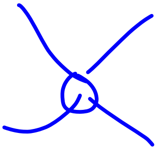

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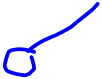

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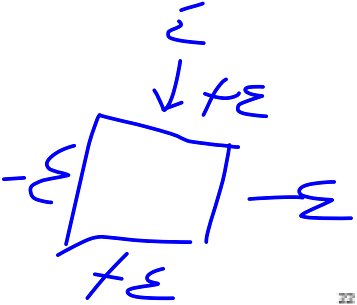
- $F(\sigma)$ has no leaves.
- We can always *fix* one edge on a cycle



matchings proof

Let $\mathcal{T}$ be a connected toast. For every $L \in \mathcal{T}_1$ there is an $m \in \omega$, an $L \subset K \in \mathcal{T}_{<m}$, and a fractional matching σ' such that

1. $\sigma'(e) \in \{0, 1\}$ for all $e \in E(L)$.
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For every $e \in L$ and $L \subset K \in \mathcal{T}$ there's a cycle in $F(\sigma)$ that's a subset of K and contains e.

Problems

- Does every one-ended bipartite Borel graph satisfy $\chi'_{BM} \leq \Delta(G)$?
- Does every bipartite d-regular Borel graph that admits a connected toast have a Borel perfect matching?